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Heaping of granular materials in a cylindrical vibrating bed

Shiu Liu and Pik-Yin Lai†

Department of Physics and Centre for Complex Systems, National Central University, Chung-li, Taiwan 320, Republic of China

E-mail: pylai@spl1.phy.ncu.edu.tw

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Abstract. We consider the heap formation of granular materials contained in a cylindrical vertically vibrating bed under slight vibration. Using the surface profile of the heap as the dynamical variable, the heap equation is generalized to the three-dimensional case. The steady state heap profiles are calculated. Our results indicate a change of downward to upward heaps as the vibration strength is increased, similar to those observed in two-dimensional vibrating bed experiments. The effective current in this model is also calculated, which can describe the convection phenomena. We also discuss the relationship between the heap formation times and the vibration strength and system size.

1. Introduction

The fact of greatest interest in studying granular materials is that they cannot be easily classified as solids, liquids or gases. Because of this unique characteristic, granular materials exhibit many unusual behaviours in a vertical vibrating bed. Many special phenomena such as size separation, pattern formation, avalanches, compaction and convection can be observed in granular systems. Granular materials behave quite differently from any of the other familiar forms of matter. For example, granular materials can sustain shears like solids with surface slope smaller than the angle of repose and also flow like liquids with slope above the angle of repose. However, as distinct from ordinary liquids, granular materials are stable as long as the top surface is at a slope less than the angle of repose. Furthermore, under periodic external excitations, granular materials can reach a steady state and can have different spatial patterns under different situations [1–12].

From past experiments, granular systems have been shown to produce some special patterns under periodic external excitations. For example, in the two-dimensional vibrating bed, we could see that as the vibration increases, the granular systems will have heaping, coherent, expansion, wave and arching formations [12]. Instead of investigating the granular materials on a microscopic particle level, in this paper we try to describe the large-scale structure of granular systems contained in a vibrating bed and analyse the steady structures and dynamics of formation of these granular systems. In this paper, we consider granular materials contained in a three-dimensional vibrating bed and the heap formation phenomena under slight vibration.

The motion of granular particles contained in a vibrating bed undergoes inelastic collisions, which causes the system energy to dissipate, but the vibrations will pump energy into the system

† Corresponding author.

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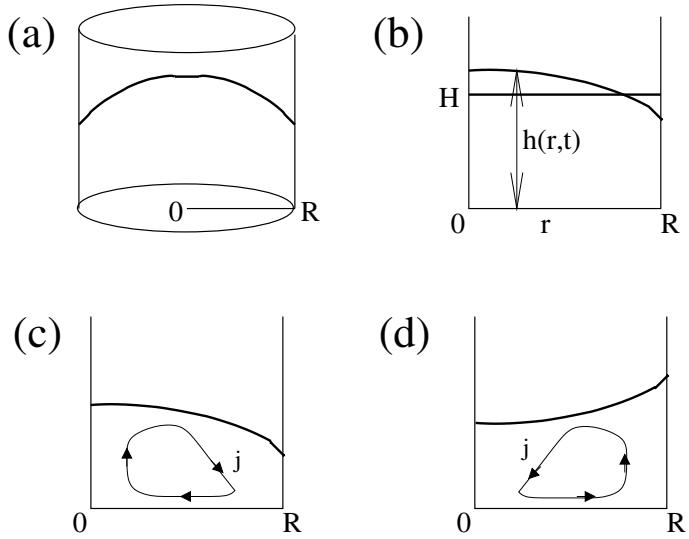

Figure 1. Schematic pictures and convection patterns in a cylindrical vibrating bed. (*a*) Cylindrical vibrating bed with origin at the centre of the base and radius R. (*b*) Initial flat layer of height H. $h(r, t)$ shows the height of pattern formation at position r and time t. (*c*) Downward heap and the convection current profile. (*d*) Upward heap and the convection roll.

continuously. The physical origin for the instability giving rise to the heap structure can be interpreted as follows: when too much energy is pumped into the system by vibration, the system cannot dissipate fast enough by small-length-scale collisions and hence large-length-scale convection (of the order of the container size) sets in at some point. The granular convection has some heuristic similarity with the Rayleigh–Benard instability, so that energy can be distributed and dissipated more efficiently.

Heap formation results from the convection as depicted schematically in figure 1. In this paper, we generalize the theoretical approach of developing a phenomenological model [13–15] to the three-dimensional vibrating bed. Using the height profile as the only dynamical variable and taking into account the decrease in local density due to vibration and the nonlinear couplings for energy dissipation, our model can reproduce the observed downward and upward heap formations. This model has been shown to describe the heap formation of the two-dimensional vibrating bed under small vibrations rather successfully; in this paper we extend our study to the three-dimensional cylindrical vibrating bed.

2. Steady state heap profile

Following the two-dimensional heap equation [13–15], our model contains energy pumped into the grains, which causes the layer to expand and the energy decrease from the collision of the grains causes the profile to flatten. We consider a cylindrical symmetric three-dimensional vibrating bed for simplicity. Let $h(\vec{r}, t)$ denote the height of the grain profile at position $\vec{r}$ and time t. $\vec{r}$ is the position vector on the base of the container and is a two-dimensional vector with radial distance r and polar angle ϕ. The equation of motion has been proposed to be [13–15]

$$\frac{\partial h}{\partial t} = D\nabla^2 h + \Omega h - \beta h^2 \tag{1}$$

where D is the diffusion constant, Ω is the coefficient of increasing height per unit time due to vibrations and β is the coefficient of decreasing height due to the dissipation of energy in the system. The centre of the circular base of the container is taken to be the origin (figure 1(a)). We assume that the initial condition is a flat profile for convenience (figure 1(b)). Therefore the initial condition is

$$h(\vec{r}, 0) = H = \text{constant}. \tag{2}$$

Since the initial profile is symmetric, and the condition of the wall is assumed to be cylindrically symmetric and will not change with time, $h(\vec{r}, t)$ will remain symmetric at later times, i.e. h is assumed to be independent of the polar angle ϕ. Therefore, one of the boundary conditions is

$$h(\vec{r}, t) = h(r, t). \tag{3}$$

Furthermore, based on the observation that the total volume of the layer is conserved for heap phenomena under small vibrations, another boundary condition is

$$\int_0^R h(r, t) r \, \mathrm{d}r = \int_0^R h(r, 0) r \, \mathrm{d}r = \frac{R^2 H}{2} \tag{4}$$

where R is the radius of the container. The steady state profile for the simple linear case when $\beta = 0$ can be solved exactly. In this case, equation (1) becomes the linear diffusion type equation

$$\frac{\partial h}{\partial t} = D\nabla^2 h + \Omega h. \tag{5}$$

Then with the boundary conditions (3) and (4), the steady state solution is solved to be

$$h_s(r) = \frac{kRH J_0(kr)}{2J_1(kR)} \tag{6}$$

where $k^2 \equiv \frac{\Omega}{D}$, and J_0, J_1 are the Bessel functions. This solution is physically valid for small enough vibration $k < \frac{x_{01}}{R}$, where $x_{01} \simeq 2.0405$ is the first root of J_0; otherwise h_s would be negative in some region of r. The dynamics for the nonlinear $\beta \neq 0$ case can be solved numerically and the results are presented in the next section. The structure of steady heaps produced from equation (1) is given by $\partial h/\partial t = 0$ and the heaping equation becomes

$$\nabla^2 h_s + k^2 h_s - \beta' h_s^2 = 0 \tag{7}$$

where $\beta' \equiv \frac{\beta}{D}$. With the cylindrical symmetric boundary condition (3), equation (7) reduces to

$$\frac{\mathrm{d}^2 h_s}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}h_s}{\mathrm{d}r} + k^2 h_s - \beta' h_s = 0. \tag{8}$$

Since all physical quantities must behave well near $r = 0$, one must have the condition $\mathrm{d}h_s(0)/\mathrm{d}r = 0$. Equation (8) can be solved numerically with the boundary conditions (4) and $\mathrm{d}h_s(0)/\mathrm{d}r = 0$. As the vibration becomes stronger, the nonlinearity becomes more important. Thus β' should increase with k. From dimensional analysis, β' must be of the form μk^3, where μ is a dimensionless parameter controlling the strength of the leading nonlinear effect.

The numerical result of h_s is shown in figure 2, where the aspect ratio is defined as $\chi = H/(2R)$. The steady state heap changes from downward (mountain) modes to upward (valley) modes as k increases. A similar behaviour has been observed in both experiments [10] and theory [13–15] in two-dimensional vibrating beds.

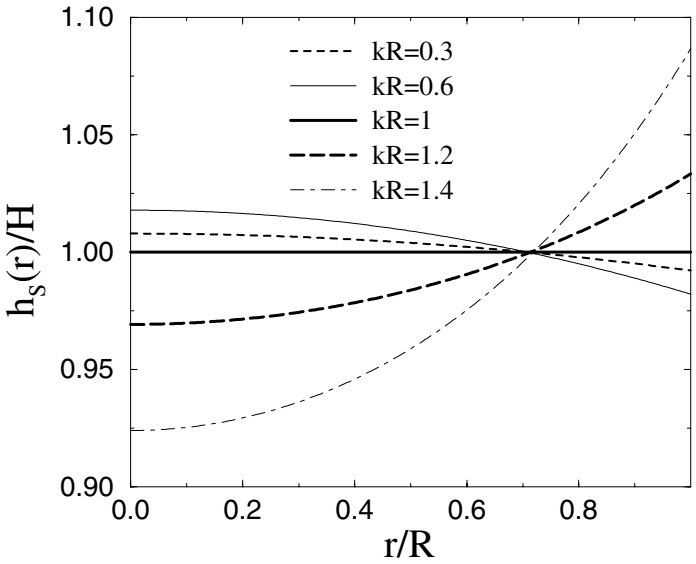

Figure 2. The steady state heap profile for different vibrating strengths kR with the same $\chi = 0.5$ and $\mu = 1$.

3. Current and dynamics of heap formation

The formation dynamics of the heap is obtained by numerically solving equation (1) with initial condition (2) and boundary conditions (3) and (4). It can be easily verified from equation (1) that, if t is in units of Ω^{-1}, there are only three independent parameters in our model, namely k, μ and the aspect ratio of the initial layer, $\chi \equiv \frac{H}{2R}$. Figure 3 shows the time evolution of the profile at a given vibrational strength. The steady state heap is downward (mountain-like) for smaller value of kR and becomes an upward (valley-like) heap at higher values of kR. It should be noted that in general the formation time is longer for stronger kR.

The conservation law given in equation (4) is equivalent to the existence of an effective current formally defined as

$$\vec{j}(\vec{r}, t) \equiv -D\nabla h - \Omega \int^{\vec{r}} h(\vec{r}', t)\, \mathrm{d}\vec{r}' + \beta \int^{\vec{r}} h^2(\vec{r}', t)\, \mathrm{d}\vec{r}' \tag{9}$$

and the heap equation (1) can be written as the continuity equation

$$\frac{\partial h}{\partial t} + \nabla \cdot \vec{j} = 0. \tag{10}$$

Because of the boundary condition of radial symmetry in h in equation (3), $\vec{j} = j_r(r, t)\hat{r}$ has only the radial component and is a function of r. The current is an odd function of r since h is even in r; furthermore, the current is equal to zero on the boundary. The first term of the current in equation (9) is the usual local gradient term, which depends only on the profile on the surface; the second and third terms depend on the bulk height of the heap. Thus the first term is related to the surface flow, while the second and third terms are dominated by the bulk flow under the surface profile. The increase in height due to vibration tries to expand the volume of the layer, but the constant volume conservation forbids such a change and the current sets in to rearrange the profile to meet the conservation law in equation (4). This current $j_r(r, t)$ can be identified as an effective current associated with the real convection current. The current

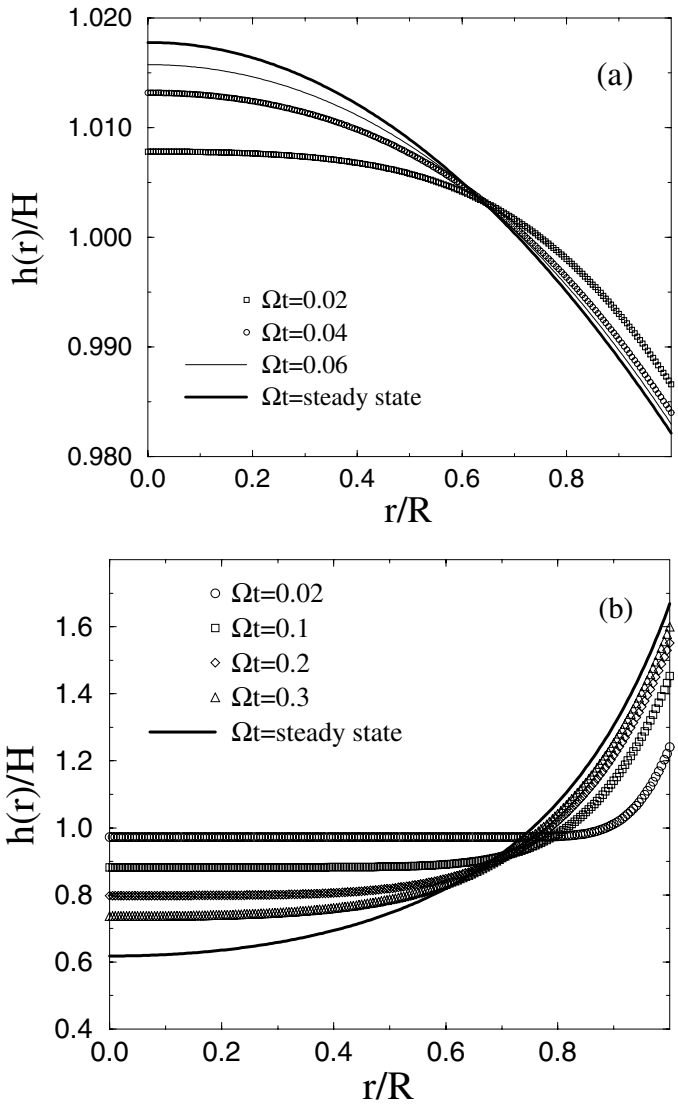

Figure 3. Time evolution of the profiles with initial flat profile with $\chi = 0.5$, $\mu = 1$. (*a*) $k = 0.6$ (*b*) $k = 2.4$.

at any time t, given by equation (9), can be computed numerically. Figure 4 displays the time evolution of the current distribution. The steady profile is a downward heap for smaller kR (figures 3(*a*) and 4(*a*)); the current flows towards the centre of the system. The current eventually goes to zero in the long-time limit. From equation (9), the current is composed of a surface current (first term in (9)) and a bulk current (second and third terms in (9)). For the downward heap case, since the surface current is positive, but the total current $j(r, t)$ is negative (see figure 4(*a*)), the bulk current must be negative deep inside the layer. Hence a convection flow pattern under the heap profile can be depicted as shown in figure 1(*c*). For larger values of kR, the steady heap is an upward mode; a similar argument (see figures 3(*b*) and 4(*b*)) would lead to the schematic flow pattern in figure 1(*d*). Although our model cannot

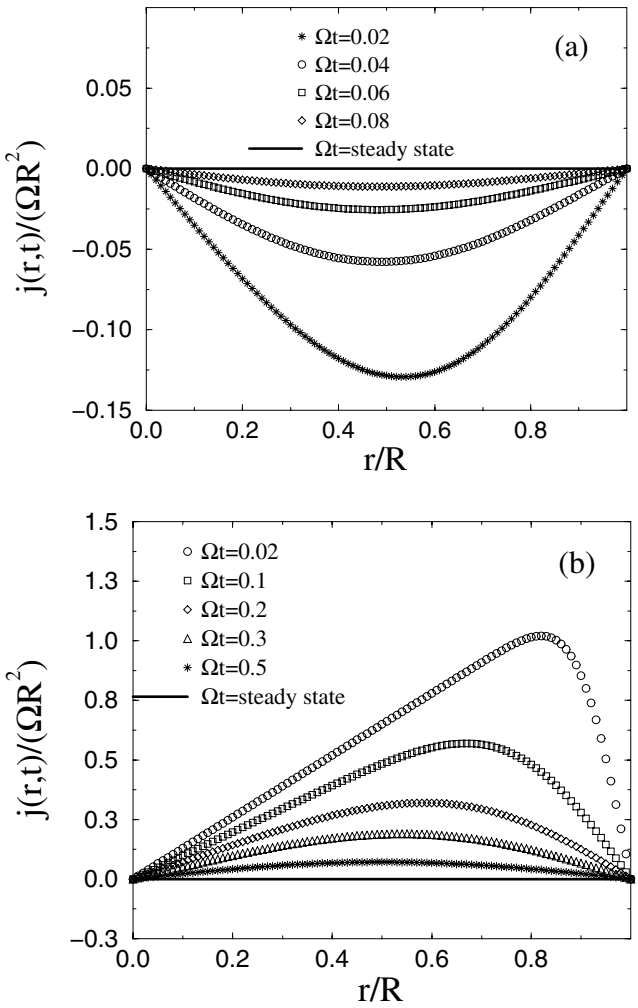

Figure 4. Time evolution of the current distributions, in units of ΩR^2, with initial flat profile and $\chi = 0.5$, $\mu = 1$: (*a*) $k = 0.6$; (*b*) $k = 2.4$.

directly calculate the detailed flow pattern inside the heaps, it does lead to a picture of the convection pattern as observed in experiments.

4. Heap formation times

We define the relaxation function $\delta(t)$ as the normalized root-mean-square deviation from the steady state profile to describe the relaxation towards the steady state as

$$[\delta(t)]^2 = \frac{\int_0^R r\,\mathrm{d}r\,[h(r,t) - h_s(r)]^2}{\int_0^R r\,\mathrm{d}r\,[h(r,0) - h_s(r)]^2}. \tag{11}$$

Obviously, $\delta(0) = 1$ and $\delta(\infty) = 0$. $\delta(t)$ obtained from the numerical solution of $h(r,t)$ for various vibrational strengths are calculated explicitly. In contrast to the results in two dimensions [14, 15], $\delta(t)$ deviates appreciably from a simple exponential decay, but can be

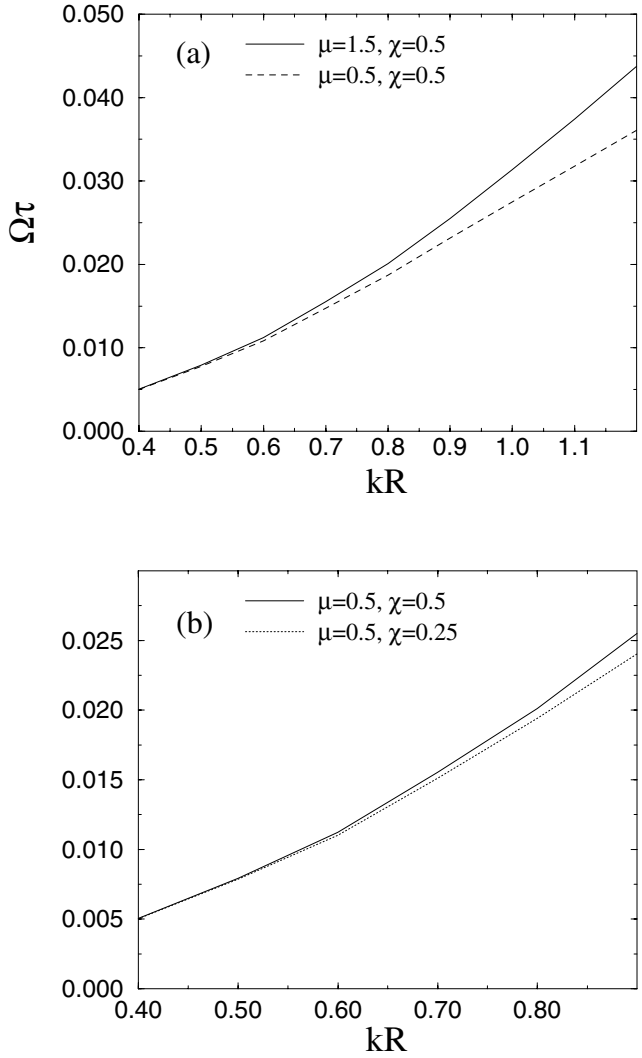

Figure 5. The heap formation times τ, in units of Ω^{-1}, as a function of the vibrating strength kR for (a) different values of μ but the same $\chi = 0.5$ and (b) different values of χ but the same $\mu = 0.5$.

fitted well by a stretched exponential function. By fitting to a stretched exponential form for t in units of Ω^{-1}, $\delta(t) = \exp(-(\frac{t}{\tilde{\tau}})^{\alpha})$, the characteristic formation time for the steady heap can be obtained. $\tilde{\tau}$ and α are fitting constants. In most cases, we find $\alpha \approx 0.94$, and $\tilde{\tau}$ increases with kR for all μ and χ. The formation times is calculated as $\tau \equiv \int_0^{\infty} \delta(t)\,\mathrm{d}t = \frac{\tilde{\tau}}{\alpha}\Gamma(\frac{1}{\alpha})$. τ (also in units of Ω^{-1}), as a function of the vibration strength kR, are shown in figure 5(a) for different values of μ. The formation time of the steady heap is longer for stronger vibration, which agrees with the general expectation that when more energy is pumped into the system, stronger fluctuations will delay the system relaxing to its steady state. Also, the formation time is shorter for a larger value of μ, which reflects the general behaviour of reaching the steady state faster for stronger dissipation that damps the energy in a dissipative system.

The formation time is also controlled by the aspect ratio $\chi = H/(2R)$ of the initial layer as shown in figure 5(*b*). For a given value of kR, a system with a smaller aspect ratio is faster in approaching the steady heap. For a system with the same R, a thicker initial layer would have a stronger dissipation relative to the input energy because the dissipative term $\sim h^2$ while the energy input term $\sim h$ (see equation (1)). The larger dissipation in a thicker layer would damp the system more effectively and form the steady heap faster.

5. Discussions

In the experimental studies [10] of heaping in two dimensions, it has been shown that two types of heap can be formed, the upward (valley-like) and the downward (mountain-like) modes. The main difference between these two modes is the convection current of the granular particles next to the wall which move up in the upward mode and move down in the downward mode. The relevant dimensionless parameter is $\Gamma = \frac{A\omega^2}{g}$, where A, ω and g are the amplitude, frequency of the vibrating bed and the gravitational acceleration respectively. It is observed that as Γ increases, the upward mode is formed and then changed to the downward mode. An originally flat layer will turn into a heap when Γ is greater than some threshold value $\Gamma_c \simeq 1.2$. The granular particles in the vibrating bed are not simply moving up and down vertically, but convection rolls with particles moving up along the wall and flows down the slope of the surface. It should be noted that the heap equation (1) is aimed at modelling the condition for heap formation, i.e. $\Gamma > \Gamma_c$ in experimental situations. For $\Gamma < \Gamma_c$, the initial flat layer remains stable. To incorporate this, the heap equation can be generalized to

$$\frac{\partial h}{\partial t} = D\nabla^2 h + \Theta(\Omega - \Omega_c) f(h) \tag{12}$$

where $\Theta(\Omega - \Omega_c)$ is the Heaviside step function that takes the values of zero or unity for the coefficient of increasing height due to vibrations Ω less or not less than the threshold value Ω_c. $\Omega > \Omega_c$ would correspond to the condition of heap formation $\Gamma > \Gamma_c$. For $\Omega < \Omega_c$, the flat layer ($h(r, t) =$ constant) will be a stable solution. $f(h)$ are some general effective nonlinear couplings. When $\Gamma \gg \Gamma_c$, the heap will disappear and the system becomes gas like. This kind of system is characterized by how the granular flow interacts with the external excitations rather than how energy is being dissipated among grain collisions. Our model concerns the small vibrations of a vibrating bed which have steady heap formation. Comparing the results of our simulation with the experiments, the vibrating strength kR, as the reduced acceleration amplitude Γ, is the most important parameter affecting the granular systems under periodic external excitations. First, the formation pattern changed from downward heaps to upward heaps as kR increased. Second, the heap formation time is a function of kR with the controlled parameters μ and χ. Our model also introduced the surface and bulk currents to represent the convection and build up the heaps, which agree with the phenomena observed from the experiments. The downward convection generates the mountain-like heaps and the upward convection produces the valley-like heaps. At even stronger vibrations, there are still other interesting patterns such as coherence, expansion, wave and arching observed in the two-dimensional vibrating bed and stripes, along with squares and hexagons in the three-dimensional thin vibrating bed [11]. The theories of these interesting phenomena are yet to be investigated.

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